



**Stochastic modeling of the risk factors of the interest**

**term structure**

**using single factor models**

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# Introduction

The term structure of interest rates is the yield curve which represents the yield of debt securities considered as default-free relative to their time to maturity. Most notably, it contains premiums accounting for the variability and expectations of future inflation and future spot rates (term premium). Default-free securities are generally very liquid, therefore, the liquidity premium is negligible.

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| M | 1 Mo | 3 Mo | 5 Mo | 1 Yr | 2 Yr | 3 Yr | 5 Yr | 7 Yr | 10 Yr | 20 Yr | 30 Yr |
| R | 1.14% | 1.27% | 1.45% | 1.62% | 1.78% | 1.90% | 2.13% | 2.28% | 2.37% | 2.58% | 2.76% |

*Data source:* [*U.S. Department of the Treasury*](https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldYear&year=2017)

An explanation of the term structure should provide a method of prediction or fitting which establishes a connection between interest rates of bonds with different maturities.

Robert Merton (1973) was the first one to propose a single factor model for the term structure of interest rates. Due to modelling the evolution of the spot rate as geometric Brownian motion, Merton’s model allows for negative interest rates, assumes that volatility and the risk premium are constant. This model serves as base for subsequent more complex single factor models.

Later on, multi-factor models have been proposed which aim to improve modelling of the term structure of interest rates by introducing factors such as stochastic drift, instantaneous inflation rate, long term rate or volatility of some sort.

There are 3 main theories which aim to explain the relationship between interest rates of risk-free bonds with various maturities.

* **Expectation hypothesis** – Investors’ expectations of future spot rates determine the term structure. Bond pricing is done on the principle that the implied forward rate should be an unbiased estimator of the future prevailing spot rate. A few variations of this hypothesis exist.
* **Liquidity preference theory** – Investors are risk-averse therefore have a preference for short-term bonds. On the other hand, long-term ones are preferred by borrowers. As result, a premium is incorporated in the price depending on the time to maturity. An important consequence is that the expected return from a buy and hold strategy will be higher than the expected return for a roll-over strategy. The resulting term structure of interest rates should be upward sloping.
* **Preferred habitat theory** – Market participants are assumed to have individual preferences regarding time to maturity. They are willing to change their “habitat” range if a sufficient premium is offered. Depending on market conditions, the risk premia associated with bonds of various maturities can be positive, negative or none. Therefore, the term structure of interest rates can take any shape.

Only the risk-neutral versions of the 3 models are presented in this paper.

# Oldrich Vasicek – An Equilibrium Characterization of the Term Structure

In the Vasicek model, the evolution of spot interest rates is characterized by a stochastic mean-reversion behavior towards a fundamental long-term value. This can be considered as theoretically as consistent with equilibrium economic models. In reality, high interest rates impede economic activity while low interest rates reduce savings and investment. Assuming a reasonable fundamental value is congruent with reality no matter how it is derived.

More precisely, the Vasicek model is an application of the Ornstein-Uhlenbeck stochastic process which

is an asymptotically stationary and distributed around a long term value.

There are the 3 hypotheses underlying the model:

* The spot interest rate follows a diffusion process
* The price of a discount bond depends only on the spot rate over its term
* The market is efficient (there is no possibility of arbitrage)

The Vasicek model assumes that the spot interest rate behaves like a first-order autoregressive process:

The latter equation could be rewritten as:

Therefore, the evolution of the spot interest is thus determined by the following diffusion process:

This form is most often used to refer to the Vasicek model.

Where:

* – convergence speed factor
* – convergence limit (fundamental value of the spot rate; could be time dependent - )
* – instantaneous volatility or randomness impacting the evolution of the interest rate
* – Wiener process
* – expected instantaneous change of the spot rate
* – exogenous shocks impacting the spot rate

Another transformed form of the model is:

This one makes easy to obtain the expected value and the variance of the process which are:

Where

The process is asymptotically stationary around the fundamental value . The long-term variance shows that and partially counteract each other’s effect. Increase in represents a faster convergence speed which could offset a non-proportional increase in .

The biggest drawback of this model is that the interest rates may become negative. This tendency is considered to be unrealistic nor desirable in most cases.

**Simulation of the model and pricing of a zero-coupon bond using R**

In order to perform simulations of the Vasicek model, equation can be approximated in discrete terms by:

The price of a zero-coupon bond with maturity at time can be obtained by solving:

Where:

The expected rate of return on any bond in excess of the spot rate is proportional to its standard deviation.

# John Cox, Jonathan Ingersoll, Stephen Ross – A Theory of the Term Structure of Interest Rates

The Cox-Ingersoll-Ross model is an extension of the Vasicek model. Unlike Vasicek, the Cox-Ingersoll-Ross (CIR) model is based on an intertemporal general equilibrium model (meaning that it is derived from a hypothetical structure of the economy).

The derived diffusion process of the spot interest rate is:

The drift factors remains the same compared to the Vasicek model, but the diffusion parameter is different. The addition of results in larger deviations when the spot rate is high, thus increasing the variance of the spot rate and in lower deviation when the spot rate is low, thus reducing the variance of the spot rate. The intuition would be that spot rates on the market are very low, volatility tends to be also very low.

The will never become negative if the initial spot interest rate is positive and because the upward drift would be bigger than a negative shock. This is attributed to the mean-reverting drift that tends to pull towards the long-run average as well as to the diminishing volatility due to being closer to zero.

The value of the spot rate at time is given by:

The expected value of the spot rate is the same as Vasicek due to the mean-reversion behavior. The variance is obviously modified due to the additional of . The asymptotic value of the first two moments remain the same.

**Simulation of the model and pricing of a zero-coupon bond using R**

In order to perform simulations of the CIR model, equation can be approximated in discrete terms by:

The price of a zero-coupon bond with maturity at time can be obtained by solving:

Where:

CIR european call on bond – analytic solution

# John Hull, Alan White – Pricing Interest-Rate-Derivative Securities

John Hull and Alan White (HW) begin their article with a generalization of the diffusion process of the spot interest rate of the Vasicek and CIR models:

* Vasicek:
* CIR:

In these versions of the models, it is assumed that the term premiums (measuring the rise of the expected return for one unit of risk) denoted as are equal to 1 (therefore ). This assumption will be used for the rest of this paper.

Ho and Lee (1986) pioneered an approach to make an interest-rate model consistent with any specified initial term structure. The HW models represent a mix of the ideas from the continuous version of the Ho and Lee model with the Vasicek and CIR models. The general form of the HW models is:

Where the additions to the Vasicek/CIR model are:

* – time-dependent drift
* – time-dependent convergence speed factor
* – time-dependent volatility factor

It is reasonable to conjecture that in some situations the market’s expectations about future interest rates involve time-dependent parameters. In other words, the drift rates and volatility of may be functions of time as well as being functions of and other state variables. The time dependence can arise from the cyclical nature of the economy, expectations concerning the future impact of monetary policies, and expected trends in other macroeconomic variables.

In this form of the model, a drift rate is imposed to the otherwise converging dynamic.

The generalized form can be rewritten as:

In this form of the model, the long-term level is a function of time.

We are going the consider the one-factor version of the HW model where only is time dependent and and are constants.

**Model 1: Extended Vasicek model (manque pas un rt dans la derniere partie de l’eq ? )**

**Model 2: Extended CIR Model**

The addition of In the One-Factor Hull-White model, this would be equivalent to have a bigger mean reversion and a smaller . From the expression of ,we see that the bigger the mean reversion, the lower the variance or .

Their disadvantage is that they involve several unobservable parameters and do no provide a perfect fit to the initial term structure of interest rates.

The volatility of the short-term interest rate can be a function of time.

Term structure of spot rate volatilities or of forward rate volatilities

The parameters of the extended Vasicek model can be determined analytically, CIR – numerically

Sigma – variance per unit time (annual)

Time dependence of the drift (& the reversion rate a) and the volatility of r – cyclicity of the economy, expectations of monetary policy

– a drift rate is imposed to an otherwise mean-reverting variable

Faire des statistiques comparatives – on fait bouger les paramètres afin de voir leur impact sur la dynamique

Add which hypothesis each of the models are based on

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